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Second order duality for nondifferentiable minimax programming problems with generalized convexity

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Abstract In this paper, we are concerned with a class of nondifferentiable minimax programming problem and its two types of second order dual models. Weak, strong and strict converse duality theorems from a view point of generalized convexity are established. Our study naturally unifies and extends some previously known results on minimax programming.

Keywords Nondifferentiable programming · Minimax programming · Second order duality · Generalized convexity

1 Introduction

Von-Neumann's classical work on games of strategy initially motivated many researchers to study minimax programming problem, which itself plays a critical role in optimization theory. Many minimax problems often arise in optimal control, engineering design, computer aided design and electronic circuit design. For many interesting applications of minimax problems such as, in the field of combinatorial optimization, problems of scheduling, allocation, packing and searching, we refer [10] and the references cited therein.

Schmitendorf [23] considered the following minimax programming problem:

$$
\begin{aligned}
(\textbf{NP})\ \text{Minimize}\quad & \sup_{y\in Y} f(x, y)\\
\text{subject to}\quad & g(x) \leq 0,\ x \in X,
\end{aligned}
$$

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where $f(\cdot,\cdot) : \mathbb{R}^n \times \mathbb{R}^l \to \mathbb{R}$ is a continuous function with continuous derivatives with respect to x, Y is a compact subset of $\mathbb{R}^l$, X is an open subset of $\mathbb{R}^n$ and $g(\cdot) : \mathbb{R}^n \to \mathbb{R}^m$ is a convex and differentiable function. The objective function above is of the type [9].

Schmitendorf [23] first established the necessary and sufficient optimality conditions for (NP) under convexity assumptions. Tanimoto [24] applied these optimality conditions to define a first order dual for minimax programming problem and derived appropriate duality results. Bector and Bhatia [5] and Weir [25] relaxed the convexity assumptions in Schmitendorf's sufficient optimality conditions; employed the optimality conditions to construct several dual problems which involve pseudoconvex and quasiconvex functions; and discussed weak and strong duality theorems. In [28], Zalmai used an infinite dimensional version of Gordan's theorem of the alternative to derive first and second order necessary optimality conditions for a class of minimax programming problems in a Banach space, and established several sufficient optimality conditions and duality formulations under generalized invexity assumptions. Mishra [17] and Bector et al. [7] have also shown their interest in developing optimality conditions and duality results for (NP). For a more comprehensive study of mathematical theory of optimization, the reader may consult [11].

Mangasarian [16] first formulated a second order dual for nonlinear programming problem and established duality results under somewhat involved assumptions. Mond [19] reproved second order duality results involving simpler assumptions than those previously taken by Mangasarian [16], and showed that the second order dual has computational advantages over the first order dual.

Bector et al. [6] first formulated several second order dual models for (NP) and derived duality results under generalized invexity, whereas Liu [15] discussed second order duality results for a general Mond–Weir type dual involving generalized B-invexity. Recently, Mishra and Rueda [18] and Ahmad et al. [4] proved second order duality theorems using generalized type-I functions for two types of dual models of a nondifferentiable minimax programming problem consisting of minimizing the supremum of a function involving square root term.

Based upon the ideas of Bector et al. [8] and Rueda et al. [22], Yang and Hou [26] proposed a new concept of generalized convexity. In this paper, we use second order extension of generalized convexity assumptions of Yang and Hou [26], to discuss duality results for Mangasarian type and general Mond–Weir type second order duals of nondifferentiable minimax programming problem [4,18].

2 Notations and preliminaries

We consider the following nondifferentiable minimax programming problem:

$$\begin{aligned} \text{(P) Minimize} \quad & \sup_{y\in Y} f(x,y) + \left(x^T Bx\right)^{\frac{1}{2}} \\ \text{subject to} \quad & g(x) \le 0,\ x \in X, \end{aligned}$$

where Y is a compact subset of $\mathbb{R}^l$, X is an open subset of $\mathbb{R}^n$; $f(\cdot,\cdot) : X \times Y \to \mathbb{R}$, and $g(\cdot) : X \to \mathbb{R}^m$ are twice differentiable functions at $x \in X$, and B is an $n \times n$ positive semidefinite symmetric matrix. If $B = 0$, then (P) is a usual minimax programming problem, which was studied by Bector and Bhatia [5], Liu [15], Schmitendorf [23], Tanimoto [24], and Weir [25].

Let $S = \{x \in X : g(x) \le 0\}$ denote the set of all feasible solutions of (P). Any point $x \in S$ is called the feasible point of (P). The index set is $M = \{1, 2, \ldots, m\}$. For each

$(x, y) \in S \times Y$, we define

$$J(x) = \left\{ j \in M : g_j(x) = 0 \right\},$$
$$Y(x) = \left\{ y \in Y : f(x, y) + \left(x^T Bx\right)^{\frac{1}{2}} = \sup_{z \in Y} f(x, z) + \left(x^T Bx\right)^{\frac{1}{2}} \right\},$$

and

$$K(x) = \left\{ (s, t, \tilde{y}) \in \mathbb{N} \times \mathbb{R}^s_+ \times \mathbb{R}^{ls} : 1 \leq s \leq n+1, t = (t_1, t_2, \ldots, t_s) \in \mathbb{R}^s_+ \right.$$
$$\left. \text{with} \sum_{i=1}^{s} t_i = 1, \tilde{y} = (\bar{y}_1, \bar{y}_2, \ldots, \bar{y}_s) \text{ with } \bar{y}_i \in Y(x), i = 1, 2, \ldots, s \right\}.$$

Definition 2.1 A functional $\mathcal{F} : X \times X \times \mathbb{R}^n \to \mathbb{R}$ is said to be sublinear in its third argument, if $\forall\, x, \bar{x} \in X$,

(i) $\mathcal{F}(x, \bar{x}; a_1 + a_2) \leq \mathcal{F}(x, \bar{x}; a_1) + \mathcal{F}(x, \bar{x}; a_2)\ \forall\, a_1, a_2 \in \mathbb{R}^n$,
(ii) $\mathcal{F}(x, \bar{x}; \alpha a) = \alpha \mathcal{F}(x, \bar{x}; a)\ \forall\, \alpha \in \mathbb{R}_+,\ a \in \mathbb{R}^n$.

By (ii), it is clear that $\mathcal{F}(x, \bar{x}; 0\, a) = 0$.

Lemma 2.1 (Generalized Schwartz inequality) *Let B be a positive semidefinite symmetric matrix of order n. Then, for all $x, u \in \mathbb{R}^n$,*

$$x^T Bu \leq \left(x^T Bx\right)^{\frac{1}{2}} \left(u^T Bu\right)^{\frac{1}{2}}.$$

We observe that the equality holds if $Bx = \lambda Bu$ for some $\lambda \geq 0$. Evidently, if $u^T Bu \leq 1$, we have $x^T Bu \leq \left(x^T Bx\right)^{\frac{1}{2}}$.

Following theorem is an special case of Theorem 3.1 in [13], and will be needed in the proofs of strong duality theorems. In what follows ∇ stands for the gradient vector with respect to x throughout the paper.

Theorem 2.1 (Necessary conditions) *If x^* is a solution (local or global) of problem (P) satisfying $x^{*T} Bx^* > 0$, and $\nabla g_j(x^*)$, $j \in J(x^*)$ are linearly independent, then there exist $(s, t, \tilde{y}) \in K(x^*)$, $u \in \mathbb{R}^n$, and $\mu \in \mathbb{R}^m_+$ such that*

$$\nabla \sum_{i=1}^{s} t_i f(x^*, \bar{y}_i) + Bu + \nabla \sum_{j=1}^{m} \mu_j g_j\left(x^*\right) = 0,$$
$$\sum_{j=1}^{m} \mu_j g_j\left(x^*\right) = 0,$$
$$t_i \geq 0,\ i = 1, 2, \ldots, s, \sum_{i=1}^{s} t_i = 1,$$
$$\left(x^{*T} Bx^*\right)^{\frac{1}{2}} = x^{*T} Bu,$$
$$u^T Bu \leq 1.$$

Throughout the paper, we assume that $\mathcal{F}$ is a sublinear functional. For $\beta = 1, 2, \ldots, r$, let $b, b_0, b_\beta : X \times X \to \mathbb{R}_+$, $\phi, \phi_0, \phi_\beta : \mathbb{R} \to \mathbb{R}$, ρ, ρ_0, ρ_β be real numbers, and let $\theta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$.

3 First duality model

In this section, we discuss usual duality results for the following second order Mangasarian [16] type dual of (P):

$$\text{(WD)} \quad \max_{(s,t,\tilde{y})\in K(z)} \sup_{(z,u,\mu,p)\in H_1(s,t,\tilde{y})} \sum_{i=1}^{s} t_i f(z, \bar{y}_i) + z^T Bu + \sum_{j=1}^{m} \mu_j g_j(z)$$
$$-\frac{1}{2} p^T \nabla^2 \left[\sum_{i=1}^{s} t_i f(z, \bar{y}_i) + \sum_{j=1}^{m} \mu_j g_j(z) \right] p,$$

where $H_1(s, t, \tilde{y})$ denotes the set of all $(z, u, \mu, p) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m_+ \times \mathbb{R}^n$ satisfying

$$\nabla \sum_{i=1}^{s} t_i f(z, \bar{y}_i) + Bu + \nabla \sum_{j=1}^{m} \mu_j g_j(z) + \nabla^2 \sum_{i=1}^{s} t_i f(z, \bar{y}_i)\, p + \nabla^2 \sum_{j=1}^{m} \mu_j g_j(z)\, p = 0, \quad (1)$$

$$u^T Bu \le 1. \quad (2)$$

If, for a triplet $(s, t, \tilde{y}) \in K(z)$, the set $H_1(s, t, \tilde{y}) = \emptyset$, then we define the supremum over it to be $-\infty$.

Theorem 3.1 (Weak duality) *Let x and $(z, u, \mu, s, t, \tilde{y}, p)$ be the feasible solutions of (P) and (WD), respectively. Suppose that there exist $\mathcal{F}, \theta, \phi, b$ and ρ such that*

$$b(x, z)\,\phi \left[\left(\sum_{i=1}^{s} t_i f(x, \bar{y}_i) + x^T Bu \right) - \left(\sum_{i=1}^{s} t_i f(z, \bar{y}_i) + z^T Bu + \sum_{j=1}^{m} \mu_j g_j(z) \right.\right.$$
$$\left.\left. -\frac{1}{2} p^T \nabla^2 \left(\sum_{i=1}^{s} t_i f(z, \bar{y}_i) + \sum_{j=1}^{m} \mu_j g_j(z) \right) p \right) \right] < 0$$
$$\Rightarrow \mathcal{F} \left(x, z; \nabla \sum_{i=1}^{s} t_i f(z, \bar{y}_i) + Bu + \nabla \sum_{j=1}^{m} \mu_j g_j(z) + \nabla^2 \sum_{i=1}^{s} t_i f(z, \bar{y}_i)\, p \right.$$
$$\left. + \nabla^2 \sum_{j=1}^{m} \mu_j g_j(z) p \right) < -\rho \|\theta(x, z)\|^2. \quad (3)$$

Further assume

$$a < 0 \Rightarrow \phi(a) < 0, \quad (4)$$
$$b(x, z) > 0, \quad (5)$$
$$\rho \ge 0, \quad (6)$$

then

$$\sup_{y\in Y} f(x,y) + \left(x^T Bx\right)^{\frac{1}{2}} \geq \sum_{i=1}^{s} t_i f(z,\bar{y}_i) + z^T Bu + \sum_{j=1}^{m} \mu_j g_j(z)$$
$$-\frac{1}{2} p^T \nabla^2 \left(\sum_{i=1}^{s} t_i f(z,\bar{y}_i) + \sum_{j=1}^{m} \mu_j g_j(z)\right) p.$$

Proof Suppose contrary to the result that

$$\sup_{y\in Y} f(x,y) + \left(x^T Bx\right)^{\frac{1}{2}} < \sum_{i=1}^{s} t_i f(z,\bar{y}_i) + z^T Bu + \sum_{j=1}^{m} \mu_j g_j(z)$$
$$-\frac{1}{2} p^T \nabla^2 \left(\sum_{i=1}^{s} t_i f(z,\bar{y}_i) + \sum_{j=1}^{m} \mu_j g_j(z)\right) p.$$

Thus, we have

$$f(x,\bar{y}_i) + \left(x^T Bx\right)^{\frac{1}{2}} < \sum_{i=1}^{s} t_i f(z,\bar{y}_i) + z^T Bu + \sum_{j=1}^{m} \mu_j g_j(z)$$
$$-\frac{1}{2} p^T \nabla^2 \left(\sum_{i=1}^{s} t_i f(z,\bar{y}_i) + \sum_{j=1}^{m} \mu_j g_j(z)\right) p,$$

for all $\bar{y}_i \in Y(x),\ i = 1, 2, \ldots, s$.

It follows from $t_i \geq 0,\ i = 1, 2, \ldots, s$, that

$$t_i \left[\left(f(x,\bar{y}_i) + \left(x^T Bx\right)^{\frac{1}{2}}\right) - \left(\sum_{i=1}^{s} t_i f(z,\bar{y}_i) + z^T Bu + \sum_{j=1}^{m} \mu_j g_j(z)\right.\right.$$
$$\left.\left.-\frac{1}{2} p^T \nabla^2 \left(\sum_{i=1}^{s} t_i f(z,\bar{y}_i) + \sum_{j=1}^{m} \mu_j g_j(z)\right) p\right)\right] \leq 0,$$

with at least one strict inequality, since $t = (t_1, t_2, \ldots, t_s) \neq 0$. Taking summation over i and using $\sum_{i=1}^{s} t_i = 1$, we have

$$\sum_{i=1}^{s} t_i f(x,\bar{y}_i) + \left(x^T Bx\right)^{\frac{1}{2}} < \sum_{i=1}^{s} t_i f(z,\bar{y}_i) + z^T Bu + \sum_{j=1}^{m} \mu_j g_j(z)$$
$$-\frac{1}{2} p^T \nabla^2 \left(\sum_{i=1}^{s} t_i f(z,\bar{y}_i) + \sum_{j=1}^{m} \mu_j g_j(z)\right) p.$$

By (2) and Lemma 2.1, we obtain

$$\sum_{i=1}^{s} t_i f(x,\bar{y}_i) + x^T Bu < \sum_{i=1}^{s} t_i f(z,\bar{y}_i) + z^T Bu + \sum_{j=1}^{m} \mu_j g_j(z)$$
$$-\frac{1}{2} p^T \nabla^2 \left(\sum_{i=1}^{s} t_i f(z,\bar{y}_i) + \sum_{j=1}^{m} \mu_j g_j(z)\right) p. \tag{7}$$

Using (4) and (5), it follows from (7) that

$$b(x,z)\,\phi\left[\left(\sum_{i=1}^{s} t_i f(x,\bar{y}_i)+x^T Bu\right)-\left(\sum_{i=1}^{s} t_i f(z,\bar{y}_i)+z^T Bu+\sum_{j=1}^{m}\mu_j g_j(z)\right.\right.$$
$$\left.\left.-\frac{1}{2}p^T\nabla^2\left(\sum_{i=1}^{s} t_i f(z,\bar{y}_i)+\sum_{j=1}^{m}\mu_j g_j(z)\right)p\right)\right]<0,$$

which along with (3) and (6) implies

$$\mathcal{F}\left(x,z;\nabla\sum_{i=1}^{s} t_i f(z,\bar{y}_i)+Bu+\nabla\sum_{j=1}^{m}\mu_j g_j(z)\right.$$
$$\left.+\nabla^2\sum_{i=1}^{s} t_i f(z,\bar{y}_i)\,p+\nabla^2\sum_{j=1}^{m}\mu_j g_j(z)\,p\right)<0,$$

which contradicts (1), since $\mathcal{F}(x,z;0)=0$. This completes the proof. □

Theorem 3.2 (Strong duality) *Assume that x^* is an optimal solution of (P) and $\nabla g_j(x^*)$, $j\in J(x^*)$ are linearly independent. Then there exist $(s^*,t^*,\tilde{y}^*)\in K(x^*)$ and $(x^*,u^*,\mu^*,p^*=0)\in H_1(s^*,t^*,\tilde{y}^*)$ such that $(x^*,u^*,\mu^*,s^*,t^*,\tilde{y}^*,p^*=0)$ is a feasible solution of (WD) and the two objectives have the same values. Further, if the hypotheses of weak duality (Theorem 3.1) hold for all feasible solutions of (P) and (WD), then $(x^*,u^*,\mu^*,s^*,t^*,\tilde{y}^*,p^*=0)$ is an optimal solution of (WD).*

Proof Since x^* is an optimal solution of (P) and $\nabla g_j(x^*)$, $j\in J(x^*)$ are linearly independent, then by Theorem 2.1, there exist $(s^*,t^*,\tilde{y}^*)\in K(x^*)$ and $(x^*,u^*,\mu^*,p^*=0)\in H_1(s^*,t^*,\tilde{y}^*)$ such that $(x^*,u^*,\mu^*,s^*,t^*,\tilde{y}^*,p^*=0)$ is a feasible solution of (WD) and the two objectives have the same values. Optimality of $(x^*,u^*,\mu^*,s^*,t^*,\tilde{y}^*,p^*=0)$ for (WD) thus follows from weak duality (Theorem 3.1). □

Theorem 3.3 (Strict converse duality) *Let x^* and $(z^*,u^*,\mu^*,s^*,t^*,\tilde{y}^*,p^*)$ be the optimal solutions of (P) and (WD), respectively. Suppose that $\nabla g_j(x^*)$, $j\in J(x^*)$ are linearly independent and there exist $\mathcal{F},\theta,\phi,b$ and ρ such that*

$$b\left(x^*,z^*\right)\phi\left[\left(\sum_{i=1}^{s^*} t_i^* f\left(x^*,\bar{y}_i^*\right)+x^{*T}Bu^*\right)-\left(\sum_{i=1}^{s^*} t_i^* f\left(z^*,\bar{y}_i^*\right)+z^{*T}Bu^*\right.\right.$$
$$\left.\left.+\sum_{j=1}^{m}\mu_j^* g_j\left(z^*\right)-\frac{1}{2}p^{*T}\nabla^2\left(\sum_{i=1}^{s^*} t_i^* f\left(z^*,\bar{y}_i^*\right)+\sum_{j=1}^{m}\mu_j^* g_j\left(z^*\right)\right)p^*\right)\right]\leq 0$$
$$\Rightarrow\mathcal{F}\left(x^*,z^*;\nabla\sum_{i=1}^{s^*} t_i^* f\left(z^*,\bar{y}_i^*\right)+Bu^*+\nabla\sum_{j=1}^{m}\mu_j^* g_j\left(z^*\right)\right.$$
$$\left.+\nabla^2\sum_{i=1}^{s^*} t_i^* f\left(z^*,\bar{y}_i^*\right)p^*+\nabla^2\sum_{j=1}^{m}\mu_j^* g_j\left(z^*\right)p^*\right)<-\rho\|\theta\left(x^*,z^*\right)\|^2. \quad (8)$$

Further assume

$$a < 0 \Rightarrow \phi(a) \leq 0, \tag{9}$$

$$b\left(x^*, z^*\right) > 0, \tag{10}$$

$$\rho \geq 0, \tag{11}$$

then $z^* = x^*$, *that is,* z^* *is an optimal solution of (P).*

Proof Suppose contrary to the result that $z^* \neq x^*$. Since x^* and $(z^*, u^*, \mu^*, s^*, t^*, \tilde{y}^*, p^*)$ are the optimal solutions of (P) and (WD), respectively, and $\nabla g_j\left(x^*\right),\ j \in J\left(x^*\right)$ are linearly independent, therefore from strong duality (Theorem 3.2), we reach at

$$\sup_{y^* \in Y} f\left(x^*, y^*\right) + \left(x^{*T} B x^*\right)^{\frac{1}{2}} = \sum_{i=1}^{s^*} t_i^* f\left(z^*, \bar{y}_i^*\right) + z^{*T} B u^* + \sum_{j=1}^{m} \mu_j^* g_j\left(z^*\right)$$
$$-\frac{1}{2} p^{*T} \nabla^2 \left(\sum_{i=1}^{s^*} t_i^* f\left(z^*, \bar{y}_i^*\right) + \sum_{j=1}^{m} \mu_j^* g_j\left(z^*\right)\right) p^*.$$

Thus, we have

$$f\left(x^*, \bar{y}_i^*\right) + \left(x^{*T} B x^*\right)^{\frac{1}{2}} \leq \sum_{i=1}^{s^*} t_i^* f\left(z^*, \bar{y}_i^*\right) + z^{*T} B u^* + \sum_{j=1}^{m} \mu_j^* g_j\left(z^*\right)$$
$$-\frac{1}{2} p^{*T} \nabla^2 \left(\sum_{i=1}^{s^*} t_i^* f\left(z^*, \bar{y}_i^*\right) + \sum_{j=1}^{m} \mu_j^* g_j\left(z^*\right)\right) p^*,$$

for all $\bar{y}_i^* \in Y\left(x^*\right),\ i = 1, 2, \ldots, s^*$.

Now proceeding as in Theorem 3.1, we get

$$\sum_{i=1}^{s^*} t_i^* f\left(x^*, \bar{y}_i^*\right) + x^{*T} B u^* < \sum_{i=1}^{s^*} t_i^* f\left(z^*, \bar{y}_i^*\right) + z^{*T} B u^* + \sum_{j=1}^{m} \mu_j^* g_j\left(z^*\right)$$
$$-\frac{1}{2} p^{*T} \nabla^2 \left(\sum_{i=1}^{s^*} t_i^* f\left(z^*, \bar{y}_i^*\right) + \sum_{j=1}^{m} \mu_j^* g_j\left(z^*\right)\right) p^*. \tag{12}$$

Using (9) and (10), it follows from (12) that

$$b\left(x^*, z^*\right) \phi \left[\left(\sum_{i=1}^{s^*} t_i^* f\left(x^*, \bar{y}_i^*\right) + x^{*T} B u^* \right) - \left(\sum_{i=1}^{s^*} t_i^* f\left(z^*, \bar{y}_i^*\right) + z^{*T} B u^* \right.\right.$$
$$\left.\left. + \sum_{j=1}^{m} \mu_j^* g_j\left(z^*\right) - \frac{1}{2} p^{*T} \nabla^2 \left(\sum_{i=1}^{s^*} t_i^* f\left(z^*, \bar{y}_i^*\right) + \sum_{j=1}^{m} \mu_j^* g_j\left(z^*\right) \right) p^* \right) \right] \leq 0,$$

which along with (8) and (11) implies

$$\mathcal{F}\left(x^*, z^*; \nabla \sum_{i=1}^{s^*} t_i^* f\left(z^*, \bar{y}_i^*\right) + Bu^* + \nabla \sum_{j=1}^{m} \mu_j^* g_j\left(z^*\right) + \nabla^2 \sum_{i=1}^{s^*} t_i^* f\left(z^*, \bar{y}_i^*\right) p^* + \nabla^2 \sum_{j=1}^{m} \mu_j^* g_j\left(z^*\right) p^*\right) < 0,$$

which contradicts (1), since $\mathcal{F}(x^*, z^*; 0) = 0$. This completes the proof. □

4 Second duality model

This section deals with the duality theorems for the following second order general Mond–Weir [21] type dual of (P):

$$\text{(MD)} \quad \max_{(s,t,\tilde{y})\in K(z)} \sup_{(z,u,\mu,p)\in H_2(s,t,\tilde{y})} \sum_{i=1}^{s} t_i f(z, \bar{y}_i) + z^T Bu + \sum_{j\in J_\circ} \mu_j g_j(z) - \frac{1}{2} p^T \nabla^2 \left(\sum_{i=1}^{s} t_i f(z, \bar{y}_i) + \sum_{j\in J_\circ} \mu_j g_j(z)\right) p,$$

where $H_2(s, t, \tilde{y})$ denotes the set of all $(z, u, \mu, p) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_+^m \times \mathbb{R}^n$ satisfying

$$\nabla \sum_{i=1}^{s} t_i f(z, \bar{y}_i) + Bu + \nabla \sum_{j=1}^{m} \mu_j g_j(z) + \nabla^2 \sum_{i=1}^{s} t_i f(z, \bar{y}_i) p + \nabla^2 \sum_{j=1}^{m} \mu_j g_j(z) p = 0, \tag{13}$$

$$\sum_{j\in J_\beta} \mu_j g_j(z) - \frac{1}{2} p^T \nabla^2 \sum_{j\in J_\beta} \mu_j g_j(z) p \geq 0, \ \beta = 1, 2, \ldots, r, \tag{14}$$

$$u^T Bu \leq 1, \tag{15}$$

where $J_\beta \in M,\ \beta = 0, 1, 2, \ldots, r$ with $\bigcup_{\beta=0}^{r} J_\beta = M$ and $J_\beta \cap J_\gamma = \emptyset$, if $\beta \neq \gamma$.

If, for a triplet $(s, t, \tilde{y}) \in K(z)$, the set $H_2(s, t, \tilde{y}) = \emptyset$, then we define the supremum over it to be $-\infty$.

Theorem 4.1 (Weak duality) *Let x and $(z, u, \mu, s, t, \tilde{y}, p)$ be the feasible solutions of (P) and (MD), respectively. Suppose that there exist $\mathcal{F}, \theta, \phi_0, b_0, \rho_0$ and $\phi_\beta, b_\beta, \rho_\beta,\ \beta = 1, 2, \ldots, r$*

such that

$$b_0(x,z)\,\phi_0\left[\left(\sum_{i=1}^{s} t_i f(x,\bar{y}_i) + x^T Bu\right) - \left(\sum_{i=1}^{s} t_i f(z,\bar{y}_i) + z^T Bu + \sum_{j\in J_0}\mu_j g_j(z)\right.\right.$$
$$\left.\left. -\frac{1}{2}p^T\nabla^2\left(\sum_{i=1}^{s} t_i f(z,\bar{y}_i) + \sum_{j\in J_0}\mu_j g_j(z)\right)p\right)\right] < 0$$
$$\Rightarrow \mathcal{F}\left(x,z;\nabla\sum_{i=1}^{s} t_i f(z,\bar{y}_i) + Bu + \nabla\sum_{j\in J_0}\mu_j g_j(z)\right.$$
$$\left. +\nabla^2\sum_{i=1}^{s} t_i f(z,\bar{y}_i)\,p + \nabla^2\sum_{j\in J_0}\mu_j g_j(z)\,p\right) < -\rho_0\|\theta(x,z)\|^2, \tag{16}$$

$$-b_\beta(x,z)\,\phi_\beta\left[\sum_{j\in J_\beta}\mu_j g_j(z) - \frac{1}{2}p^T\nabla^2\sum_{j\in J_\beta}\mu_j g_j(z)\,p\right] \leq 0$$
$$\Rightarrow \mathcal{F}\left(x,z;\nabla\sum_{j\in J_\beta}\mu_j g_j(z) + \nabla^2\sum_{j\in J_\beta}\mu_j g_j(z)\,p\right)$$
$$\leq -\rho_\beta\|\theta(x,z)\|^2,\ \beta = 1,2,\ldots,r. \tag{17}$$

Further assume

$$a \geq 0 \Rightarrow \phi_\beta(a) \geq 0,\ \beta = 1,2,\ldots,r, \tag{18}$$
$$a < 0 \Rightarrow \phi_0(a) < 0, \tag{19}$$
$$b_0(x,z) > 0,\ b_\beta(x,z) \geq 0,\ \beta = 1,2,\ldots,r, \tag{20}$$
$$\rho_0 + \sum_{\beta=1}^{r}\rho_\beta \geq 0, \tag{21}$$

then

$$\sup_{y\in Y} f(x,y) + \left(x^T Bx\right)^{\frac{1}{2}} \geq \sum_{i=1}^{s} t_i f(z,\bar{y}_i) + z^T Bu + \sum_{j\in J_\circ}\mu_j g_j(z)$$
$$-\frac{1}{2}p^T\nabla^2\left(\sum_{i=1}^{s} t_i f(z,\bar{y}_i) + \sum_{j\in J_\circ}\mu_j g_j(z)\right)p.$$

Proof Suppose contrary to the result that

$$\sup_{y\in Y} f(x,y) + \left(x^T Bx\right)^{\frac{1}{2}} < \sum_{i=1}^{s} t_i f(z,\bar{y}_i) + z^T Bu + \sum_{j\in J_\circ}\mu_j g_j(z)$$
$$-\frac{1}{2}p^T\nabla^2\left(\sum_{i=1}^{s} t_i f(z,\bar{y}_i) + \sum_{j\in J_\circ}\mu_j g_j(z)\right)p.$$

Thus, we have

$$f(x,\bar{y}_i)+\left(x^TBx\right)^{\frac{1}{2}}<\sum_{i=1}^{s}t_if(z,\bar{y}_i)+z^TBu+\sum_{j\in J_\circ}\mu_jg_j(z)$$
$$-\frac{1}{2}p^T\nabla^2\left(\sum_{i=1}^{s}t_if(z,\bar{y}_i)+\sum_{j\in J_\circ}\mu_jg_j(z)\right)p,$$

for all $\bar{y}_i\in Y(x), i=1,2,\dots,s$.

It follows from $t_i\geq 0,\ i=1,2,\dots,s$, that

$$t_i\left[\left(f(x,\bar{y}_i)+\left(x^TBx\right)^{\frac{1}{2}}\right)-\left(\sum_{i=1}^{s}t_if(z,\bar{y}_i)+z^TBu+\sum_{j\in J_\circ}\mu_jg_j(z)\right.\right.$$
$$\left.\left.-\frac{1}{2}p^T\nabla^2\left(\sum_{i=1}^{s}t_if(z,\bar{y}_i)+\sum_{j\in J_\circ}\mu_jg_j(z)\right)p\right)\right]\leq 0,$$

with at least one strict inequality, since $t=(t_1,t_2,\dots,t_s)\neq 0$. Taking summation over i and using $\sum_{i=1}^{s}t_i=1$, we have

$$\sum_{i=1}^{s}t_if(x,\bar{y}_i)+\left(x^TBx\right)^{\frac{1}{2}}<\sum_{i=1}^{s}t_if(z,\bar{y}_i)+z^TBu+\sum_{j\in J_\circ}\mu_jg_j(z)$$
$$-\frac{1}{2}p^T\nabla^2\left(\sum_{i=1}^{s}t_if(z,\bar{y}_i)+\sum_{j\in J_\circ}\mu_jg_j(z)\right)p.$$

The above inequality with Lemma 2.1 and (15) yields

$$\sum_{i=1}^{s}t_if(x,\bar{y}_i)+x^TBu<\sum_{i=1}^{s}t_if(z,\bar{y}_i)+z^TBu+\sum_{j\in J_\circ}\mu_jg_j(z)$$
$$-\frac{1}{2}p^T\nabla^2\left(\sum_{i=1}^{s}t_if(z,\bar{y}_i)+\sum_{j\in J_\circ}\mu_jg_j(z)\right)p. \tag{22}$$

Using (19) and (20), it follows from (22) that

$$b_0(x,z)\phi_0\left[\left(\sum_{i=1}^{s}t_if(x,\bar{y}_i)+x^TBu\right)-\left(\sum_{i=1}^{s}t_if(z,\bar{y}_i)+z^TBu+\sum_{j\in J_0}\mu_jg_j(z)\right.\right.$$
$$\left.\left.-\frac{1}{2}p^T\nabla^2\left(\sum_{i=1}^{s}t_if(z,\bar{y}_i)+\sum_{j\in J_0}\mu_jg_j(z)\right)p\right)\right]<0,$$

which by (16) implies

$$\mathcal{F}\left(x, z; \nabla \sum_{i=1}^{s} t_i f(z, \bar{y}_i) + Bu + \nabla \sum_{j\in J_0} \mu_j g_j(z) + \nabla^2 \sum_{i=1}^{s} t_i f(z, \bar{y}_i)\, p + \nabla^2 \sum_{j\in J_0} \mu_j g_j(z)\, p\right) < -\rho_0 \|\theta(x, z)\|^2. \tag{23}$$

Also, the inequality (14) along with (18) and (20) yields

$$-b_\beta(x, z)\, \phi_\beta\left[\sum_{j\in J_\beta} \mu_j g_j(z) - \frac{1}{2} p^T \nabla^2 \sum_{j\in J_\beta} \mu_j g_j(z)\, p\right] \leq 0, \ \ \beta = 1, 2, \ldots, r.$$

From (17) and the above inequality, we have

$$\mathcal{F}\left(x, z; \nabla \sum_{j\in J_\beta} \mu_j g_j(z) + \nabla^2 \sum_{j\in J_\beta} \mu_j g_j(z)\, p\right) \leq -\rho_\beta \|\theta(x, z)\|^2, \ \ \beta = 1, 2, \ldots, r. \tag{24}$$

On adding (23) and (24) and making use of the sublinearity of $\mathcal{F}$ with (21), we obtain

$$\mathcal{F}\left(x, z; \nabla \sum_{i=1}^{s} t_i f(z, \bar{y}_i) + Bu + \nabla \sum_{j=1}^{m} \mu_j g_j(z) + \nabla^2 \sum_{i=1}^{s} t_i f(z, \bar{y}_i)\, p + \nabla^2 \sum_{j=1}^{m} \mu_j g_j(z)\, p\right) < 0,$$

which contradicts (13), since $\mathcal{F}(x, z; 0) = 0$. This completes the proof. □

The proof of the following theorem is similar to that of Theorem 3.2 and hence, being omitted.

Theorem 4.2 (Strong duality) *Assume that x^* is an optimal solution of (P) and $\nabla g_j(x^*)$, $j \in J(x^*)$ are linearly independent. Then there exist $(s^*, t^*, \tilde{y}^*) \in K(x^*)$ and $(x^*, u^*, \mu^*, p^* = 0) \in H_2(s^*, t^*, \tilde{y}^*)$ such that $(x^*, u^*, \mu^*, s^*, t^*, \tilde{y}^*, p^* = 0)$ is a feasible solution of (MD) and the two objectives have the same values. Further, if the hypotheses of weak duality (Theorem 4.1) hold for all feasible solutions of (P) and (MD), then $(x^*, u^*, \mu^*, s^*, t^*, \tilde{y}^*, p^* = 0)$ is an optimal solution of (MD).*

Theorem 4.3 (Strict converse duality) *Let x^* and $(z^*, u^*, \mu^*, s^*, t^*, \tilde{y}^*, p^*)$ be the optimal solutions of (P) and (MD), respectively. Suppose that $\nabla g_j(x^*)$, $j \in J(x^*)$ are linearly* *independent and there exist* $\mathcal{F}, \theta, \phi_0, b_0, \rho_0$ *and* $\phi_\beta, b_\beta, \rho_\beta,\ \beta = 1, 2, \ldots, r$ *such that*

$$b_0\left(x^*, z^*\right)\phi_0\left[\left(\sum_{i=1}^{s^*} t_i^* f\left(x^*, \bar{y}_i^*\right) + x^{*T}Bu^*\right) - \left(\sum_{i=1}^{s^*} t_i^* f\left(z^*, \bar{y}_i^*\right) + z^{*T}Bu^*\right.\right.$$

$$\left.\left. + \sum_{j\in J_0}\mu_j^* g_j\left(z^*\right) - \frac{1}{2}p^{*T}\nabla^2\left(\sum_{i=1}^{s^*} t_i^* f\left(z^*, \bar{y}_i^*\right) + \sum_{j\in J_0}\mu_j^* g_j\left(z^*\right)\right)p^*\right)\right] \leq 0$$

$$\Rightarrow \mathcal{F}\left(x^*, z^*; \nabla\sum_{i=1}^{s^*} t_i^* f\left(z^*, \bar{y}_i^*\right) + Bu^* + \nabla\sum_{j\in J_0}\mu_j^* g_j\left(z^*\right)\right.$$

$$\left. + \nabla^2\sum_{i=1}^{s^*} t_i^* f\left(z^*, \bar{y}_i^*\right)p^* + \nabla^2\sum_{j\in J_0}\mu_j^* g_j\left(z^*\right)p^*\right) < -\rho_0\|\theta\left(x^*, z^*\right)\|^2, \tag{25}$$

$$-b_\beta\left(x^*, z^*\right)\phi_\beta\left[\sum_{j\in J_\beta}\mu_j^* g_j\left(z^*\right) - \frac{1}{2}p^{*T}\nabla^2\sum_{j\in J_\beta}\mu_j^* g_j\left(z^*\right)p^*\right] \leq 0$$

$$\Rightarrow \mathcal{F}\left(x^*, z^*; \nabla\sum_{j\in J_\beta}\mu_j^* g_j\left(z^*\right) + \nabla^2\sum_{j\in J_\beta}\mu_j^* g_j\left(z^*\right)p^*\right) \leq -\rho_\beta\|\theta\left(x^*, z^*\right)\|^2,$$

$$\beta = 1, 2, \ldots, r. \tag{26}$$

Further assume

$$a \geq 0 \Rightarrow \phi_\beta(a) \geq 0,\ \beta = 1, 2, \ldots, r, \tag{27}$$

$$a < 0 \Rightarrow \phi_0(a) \leq 0, \tag{28}$$

$$b_0\left(x^*, z^*\right) > 0,\ b_\beta\left(x^*, z^*\right) \geq 0,\ \beta = 1, 2, \ldots, r, \tag{29}$$

$$\rho_0 + \sum_{\beta=1}^{r}\rho_\beta \geq 0, \tag{30}$$

then $z^* = x^*$*, that is,* z^* *is an optimal solution of (P).*

Proof Suppose contrary to the result that $z^* \neq x^*$. Since x^* and $(z^*, u^*, \mu^*, s^*, t^*, \tilde{y}^*, p^*)$ are the optimal solutions of (P) and (MD), respectively, and $\nabla g_j\left(x^*\right),\ j \in J\left(x^*\right)$ are linearly independent, therefore from strong duality (Theorem 4.2), we reach at

$$\sup_{y^*\in Y} f\left(x^*, y^*\right) + \left(x^{*T}Bx^*\right)^{\frac{1}{2}} = \sum_{i=1}^{s^*} t_i^* f\left(z^*, \bar{y}_i^*\right) + z^{*T}Bu^* + \sum_{j\in J_0}\mu_j^* g_j\left(z^*\right)$$

$$-\frac{1}{2}p^{*T}\nabla^2\left(\sum_{i=1}^{s^*} t_i^* f\left(z^*, \bar{y}_i^*\right) + \sum_{j\in J_0}\mu_j^* g_j\left(z^*\right)\right)p^*.$$

Thus, we have

$$f\left(x^*, \bar{y}_i^*\right) + \left(x^{*T} B x^*\right)^{\frac{1}{2}} \leq \sum_{i=1}^{s^*} t_i^* f\left(z^*, \bar{y}_i^*\right) + z^{*T} B u^* + \sum_{j \in J_0} \mu_j^* g_j\left(z^*\right)$$
$$-\frac{1}{2} p^{*T} \nabla^2 \left(\sum_{i=1}^{s^*} t_i^* f\left(z^*, \bar{y}_i^*\right) + \sum_{j \in J_0} \mu_j^* g_j\left(z^*\right) \right) p^*,$$

for all $\bar{y}_i^* \in Y\left(x^*\right),\ i = 1, 2, \ldots, s^*$.

Now proceeding as in Theorem 4.1, we get

$$\sum_{i=1}^{s^*} t_i^* f\left(x^*, \bar{y}_i^*\right) + x^{*T} B u^* < \sum_{i=1}^{s^*} t_i^* f\left(z^*, \bar{y}_i^*\right) + z^{*T} B u^* + \sum_{j \in J_0} \mu_j^* g_j\left(z^*\right)$$
$$-\frac{1}{2} p^{*T} \nabla^2 \left(\sum_{i=1}^{s^*} t_i^* f\left(z^*, \bar{y}_i^*\right) + \sum_{j \in J_0} \mu_j^* g_j\left(z^*\right) \right) p^*. \tag{31}$$

Using (28) and (29), it follows from (31) that

$$b_0\left(x^*, z^*\right) \phi_0 \left[\left(\sum_{i=1}^{s^*} t_i^* f\left(x^*, \bar{y}_i^*\right) + x^{*T} B u^* \right) - \left(\sum_{i=1}^{s^*} t_i^* f\left(z^*, \bar{y}_i^*\right) + z^{*T} B u^* \right.\right.$$
$$\left.\left. + \sum_{j \in J_0} \mu_j^* g_j\left(z^*\right) - \frac{1}{2} p^{*T} \nabla^2 \left(\sum_{i=1}^{s^*} t_i^* f\left(z^*, \bar{y}_i^*\right) + \sum_{j \in J_0} \mu_j^* g_j\left(z^*\right) \right) p^* \right) \right] \leq 0,$$

which along with (25) implies

$$\mathcal{F}\left(x^*, z^*; \nabla \sum_{i=1}^{s^*} t_i^* f\left(z^*, \bar{y}_i^*\right) + B u^* + \nabla \sum_{j \in J_0} \mu_j^* g_j\left(z^*\right) \right.$$
$$\left. + \nabla^2 \sum_{i=1}^{s^*} t_i^* f\left(z^*, \bar{y}_i^*\right) p^* + \nabla^2 \sum_{j \in J_0} \mu_j^* g_j\left(z^*\right) p^* \right) < -\rho_0 \|\theta\left(x^*, z^*\right)\|^2. \tag{32}$$

Also, from (14), (27) and (29), we have

$$-b_\beta\left(x^*, z^*\right) \phi_\beta \left[\sum_{j \in J_\beta} \mu_j^* g_j\left(z^*\right) - \frac{1}{2} p^{*T} \nabla^2 \sum_{j \in J_\beta} \mu_j^* g_j\left(z^*\right) p^* \right] \leq 0,\ \beta = 1, 2, \ldots, r.$$

The above inequality together with (26) gives

$$\mathcal{F}\left(x^*, z^*; \nabla \sum_{j \in J_\beta} \mu_j^* g_j\left(z^*\right) + \nabla^2 \sum_{j \in J_\beta} \mu_j^* g_j\left(z^*\right) p^* \right)$$
$$\leq -\rho_\beta \|\theta\left(x^*, z^*\right)\|^2,\ \beta = 1, 2, \ldots, r. \tag{33}$$

On adding (32) and (33) and making use of the sublinearity of F and (30), we have

$$\mathcal{F}\left(x^*, z^*; \nabla \sum_{i=1}^{s^*} t_i^* f\left(z^*, \bar{y}_i^*\right) + Bu^* + \nabla \sum_{j=1}^{m} \mu_j^* g_j\left(z^*\right) + \nabla^2 \sum_{i=1}^{s^*} t_i^* f\left(z^*, \bar{y}_i^*\right) p^* + \nabla^2 \sum_{j=1}^{m} \mu_j^* g_j\left(z^*\right) p^*\right) < 0,$$

which contradicts (13), since $\mathcal{F}(x^*, z^*; 0) = 0$. This completes the proof. □

5 Special cases

- (i) In case Y is singleton, the problem (P) becomes the nondifferentiable programming problem [20,29], and corresponding duals (WD) and (MD) reduce to second order duals (2MD) and (2GMD) proposed by Zhang and Mond [29]. If, in addition $p = 0$, then we obtain the first order duals of Mond [20] and Zhang and Mond [29].
- (ii) If Y is singleton and $B = 0$, then our duals reduce to the well known second order Mangasarian [16] and general Mond–Weir [21] duals.
- (iii) Let $B = 0$. Then (WD) reduces to one of duals discussed by Bector et al. [6], and (MD) reduces to the dual formulated by Liu [15].

6 Concluding remarks

I. It is remarkable that previously known results [5,6,15,17,23,25] somehow appear as special cases of our study. It will be interesting to see whether or not the duality results developed here still hold for the following nondifferentiable problems:

Minimax fractional programming problem [2,3,13]:

$$\begin{aligned} &\text{Minimize} \quad \sup_{y\in Y} \frac{f(x, y) + \left(x^t Bx\right)^{\frac{1}{2}}}{h(x, y) - (x^t Dx)^{\frac{1}{2}}}, \\ &\text{subject to} \quad g(x) \leq 0, x \in X, \end{aligned}$$

where Y is a compact subset of $\mathbb{R}^l$, $f, h(\cdot,\cdot) : \mathbb{R}^n \times \mathbb{R}^l \to \mathbb{R}$, are C^1 functions on $\mathbb{R}^n \times \mathbb{R}^l$ and $g(\cdot) : \mathbb{R}^n \to \mathbb{R}^m$ is C^1 function on $\mathbb{R}^n$.B and D are $n \times n$ positive semidefinite symmetric matrices.

Complex minimax fractional programming problem [12]:

$$\begin{aligned} &\text{Minimize} \quad \sup_{\nu\in W} \frac{Re\left[f(\xi, \nu) + \left(z^H Bz\right)^{\frac{1}{2}}\right]}{Re\left[h(\xi, \nu) - \left(z^H Dz\right)^{\frac{1}{2}}\right]} \\ &\text{subject to} \quad -g(\xi) \in S, \ \xi \in \mathbb{C}^{2n}, \end{aligned}$$

where $\xi = (z, \bar{z}), \nu = (\omega, \bar{\omega})$ for $z \in \mathbb{C}^n, \omega \in \mathbb{C}^l$. $f(\cdot,\cdot) : \mathbb{C}^{2n} \times \mathbb{C}^{2l} \to \mathbb{C}$ and $h(\cdot,\cdot) : \mathbb{C}^{2n} \times \mathbb{C}^{2l} \to \mathbb{C}$ are analytic with respect to ξ, W is a specified compact subset in $\mathbb{C}^{2l}$, S is a polyhedral cone in $\mathbb{C}^m$ and $g : \mathbb{C}^{2n} \to \mathbb{C}^m$ is analytic. Also $B, D \in \mathbb{C}^{n\times n}$ are positive semidefinite Hermitian matrices.

II. Various considerable developments have been done for generalizing the notion of convexity, and to extend the validity of results to larger classes of optimization problems. One of the important generalizations in this connection is $(\mathcal{F}, \alpha, \rho, d)$-convexity, which was introduced by Liang et al. [14]. This concept was further extended to second order $(\mathcal{F}, \alpha, \rho, d)$-convexity by Ahmad and Husain [1]. Recently, Yuan et al. [27] proposed the concept of $(\mathcal{C}, \alpha, \rho, d)$-convexity, which includes many of the convexity concepts. One can try to prove the duality results discussed here by defining second order $(\mathcal{C}, \alpha, \rho, d)$-convexity and its generalization.

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